Control of crisis-induced intermittency in the dynamics of a kicked, damped spin

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(Received 21 November 1996)

A method of controlling the intermittent behavior induced by a crisis is applied to a model of a periodically kicked, damped spin. Using small, occasional changes of a parameter, one can make the system remain on a former attractor, enlarged or destroyed by a crisis. The amplitude of the changes grows linearly with the distance from the crisis point, and the frequency of interventions scales as the inverse of the characteristic time of the intermittency. [S1063-651X(97)05704-8]

PACS number(s): 05.45.+b

I. INTRODUCTION

Control of chaos is one of the extensively explored topics in recent years [1–3]. The generic feature of nonlinear (chaotic) dynamical systems, namely, their sensitivity to small changes of the system parameters or variables, enables obtaining significant changes in the dynamics by applying only a small control signal. One of the purposes of the control may be the stabilization of the unstable periodic orbits embedded in the chaotic attractor [1]. Another case is connected with the event called a *boundary crisis* [4–6]: the collision of the attractor with its basin of attraction, when varying a system parameter p and passing some critical value p_c. Such collision implies a sudden change in the system dynamics: (i) destruction of the attractor or (ii) merging with another attractor (or attractors). Connected with the crises is a characteristic transient or intermittent behavior appearing at the parameter values p slightly greater than the crisis value p_c (after the crisis). In case (i) the system spends some time on the remanent of the destroyed attractor before it moves to some other part of the phase space. When (ii) occurs, one can observe intermittent jumps among different parts of the attractor. The time between subsequent jumps (ii) or the transient time (i) is exponentially distributed

$$Pr(t)=\frac{1}{\tau}\exp\left(-\frac{t}{\tau}\right), \tag{1}$$

and its mean value τ scales as

$$\tau\propto(p-p_c)^{-\gamma} \tag{2}$$

for a large class of low dimensional dynamical systems, with the exponent γ being determined by the eigenvalues of the periodic orbit involved in the crisis [6].

One might be interested in making the system remain on the chaotic attractor or on a definite part of it. One way to do this [7,8] is to find an arbitrarily long chaotic orbit lying within the desired part of the attractor (or its remanent), and then control it using the classical method devised in [1].

Another approach [9], investigated in this paper, makes the use of the specific geometry of the phase space arising after the crisis. Generically, the chaotic attractor lies in the closure of the unstable manifold w_u of a periodic orbit and the boundary of its basin of attraction is in the closure of the stable manifold w_s of the same or another orbit, which we call the mediating orbit, as it is a kind of a gateway for the system to escape from the attractor. When the crisis occurs both manifolds become tangent one to another in infinitely many points (homoclinic or heteroclinic points, depending on whether w_s and w_u belong to the same or different orbits, respectively). As the parameter p is further increased, the tangency points transform into a series of regions along which the system escapes from the attractor. The main idea of the control is to break up the escape, applying a perturbation in the system.

In the present paper we develop the idea suggested in [9], giving an explicit formula for the control signal and calculating its mean value in the case of a two-dimensional (2D) map. The approach is applied to the model of a kicked classical spin.

In Sec. II, we introduce the model of a periodically kicked, damped spin. Section III describes the idea of control intermittency appearing in the model. The numerical results are given in Sec. IV. Finally, Sec. V provides a discussion of various aspects of the method, as well as a comparison to other methods of controlling transient chaotic dynamics.

II. DYNAMICS OF THE PERIODICALLY KICKED, DAMPED SPIN

After the papers [10,11], we consider a classical magnetic moment (spin) $\mathbf{S}$, $|\mathbf{S}|=S$ in the field of uniaxial anisotropy (z is the easy or hard axis) with imposed transversal magnetic field $B(t)$ along the x axis. The system can be described by the Hamiltonian

$$H=-A(S_z)^2-B(t)S_x, \tag{3}$$

where A is the anisotropy constant; we have the easy z axis when $A>0$ and the easy x-y plane (hard z axis) when $A<0$.

The motion of the spin is determined by the Landau-Lifschitz equation with damping term

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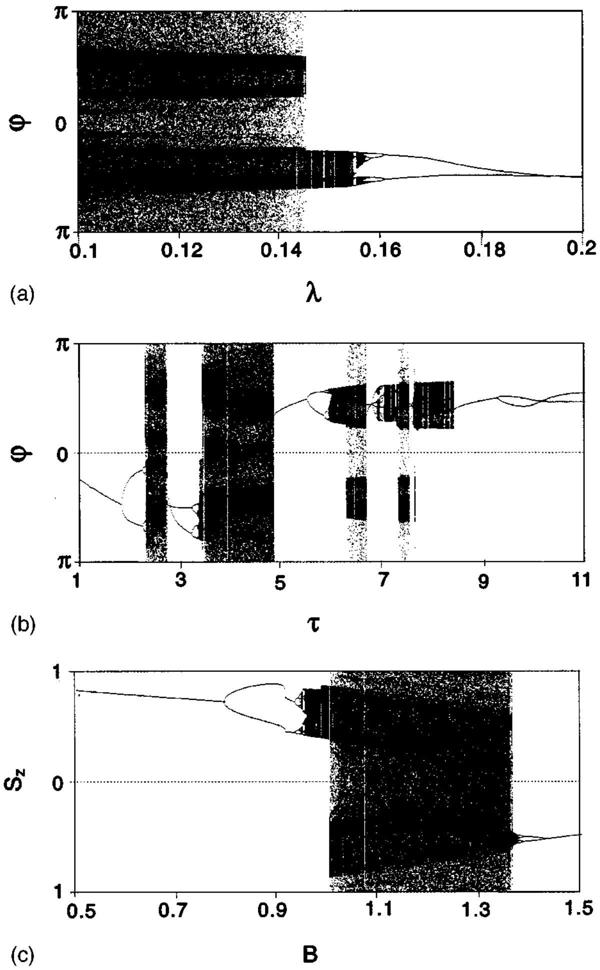

FIG. 1. Bifurcation diagrams for the spin map (8) for $A=1$ and (a) $\tau=2\pi$, $B=1$; (b) $\lambda=0.105\,494\,2$, $B=1$; (c) $\tau=2\pi,\lambda=0.105\,494\,2$.

$$\frac{d\mathbf{S}}{dt}=\mathbf{S}\times\mathbf{B}_{\text{eff}}-\frac{\lambda}{S}\mathbf{S}\times(\mathbf{S}\times\mathbf{B}_{\text{eff}}), \tag{4}$$

where $\mathbf{B}_{\text{eff}}=-dH/d\mathbf{S}$ is the effective magnetic field and $\lambda>0$ is a damping parameter.

Taking the driving field in the form of periodic δ pulses of the amplitude B and the period τ

$$B(t)=B\sum_{n=1}^{\infty}\delta(t-n\tau), \tag{5}$$

and using the fact that $|\mathbf{S}|$ is constant, the equation of motion (4) can be transformed into a superposition of two 2D maps: T_A describing the time evolution between kicks and T_B describing the effect of the kick. The map T_A can be written in the variables (S_z,φ), where φ is the angle between the axis x and the projection of the spin on the x-y plane

$$T_A\begin{bmatrix}\varphi\\ S_z\end{bmatrix}=\begin{bmatrix}\varphi+\Delta\varphi\\ WS_z\end{bmatrix}, \tag{6}$$

where $W=[c^2+(S_z/S)^2(1-c^2)]^{-1/2}$, $c=\exp(-2\lambda AS\tau)$ and $\Delta\varphi=(1/\lambda)\ln(1+S/S_z)/[1+S/(WS_z)]-2AS\tau$.

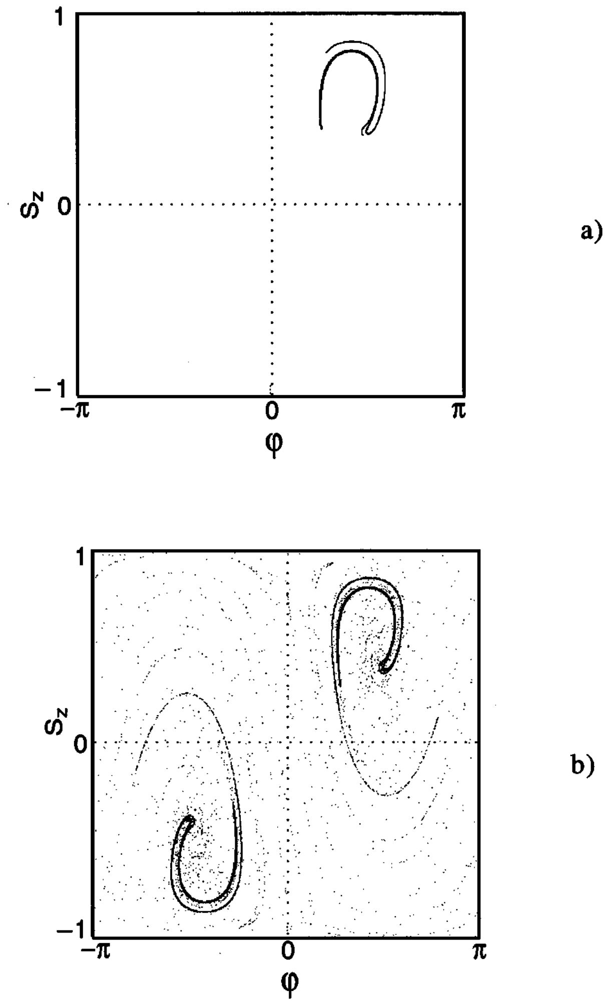

FIG. 2. (a) Chaotic attractor of the map (8) for $A_c=1$, $\tau_c=2\pi$, $\lambda_c=0.105\,494\,2$, and $B=0.999$, just before crisis [another, separate attractor is situated symmetrically with respect to the point (0,0)]. (b) Attractor arisen after merging of the formerly separate attractors after crisis (for $A_c=1$, $\lambda_c=0.105\,494\,2$, and $B=1.001$).

The second map T_B written in the variables (S_x,Φ), where Φ is the angle between the y axis and the projection of the spin on the y-z plane has the form

$$T_B\begin{bmatrix}\Phi\\ S_x\end{bmatrix}=\begin{bmatrix}\Phi-B\\ S-2S(S-S_x)D^2U\end{bmatrix}, \tag{7}$$

where $D=\exp(-\lambda B)$ and $U=[S+S_x+D^2(S-S_x)]^{-1}$.

The complete dynamics is yielded as a composition of the two maps

$$\mathbf{S}'=T_B[T_A[\mathbf{S}]] \tag{8}$$

with appropriate transformation of coordinates. The classical undamped case of the map (8) as well as its corresponding quantum model has been investigated in several papers, e.g., [12,13].

For different values of the parameters the system exhibits various types of dynamics including the periodic and chaotic ones (Fig. 1). In this paper, we study the crisis that occurs for

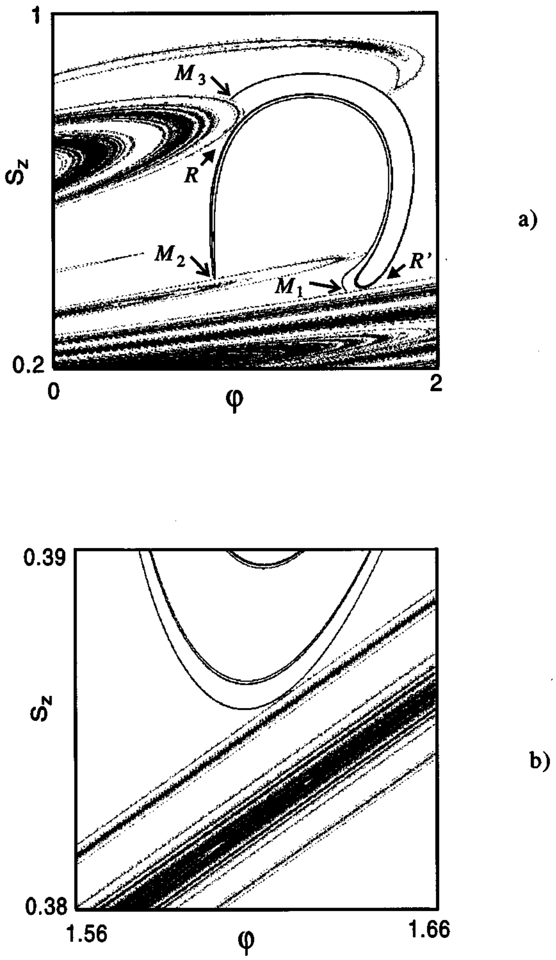

FIG. 3. (a) Chaotic attractor [compare Fig. 2(a)] and its basin of attraction (white spots) just before crisis [parameter values as in Fig. 2(a)]. Black spots denote the basin of attraction of the other, symmetric attractor. Mediating orbit (period 3) is marked by M_1, M_2, M_3. (b) A blown up tangency region from (a); the fractal structure of the basin boundary is visible.

$\tau_c = 2\pi$, $\lambda_c = 0.105\,494\,2$, $A_c = 1$, and $B_c = 1$ in which two symmetric chaotic attractors merge [10,11] (Fig. 2). The attractors correspond to two Ising states (spin ''up'' and ''down'') existing in the absence of the external field. We take the amplitude of the driving field, B, as an accessible system parameter. For $B > B_c$ random jumps between the two, previously separate attractors can be observed; the time between jumps obeys (roughly) the scaling low (2). Such a phenomenon is called a *crisis induced intermittency* [6]. Here we deal with a homoclinic crisis with the mediating saddle period 3 orbit, marked in Fig. 3. The eigenvalues of the orbit are $\lambda_u \approx 4.8988$ and $\lambda_s \approx 0.0108$ and hence, the exponent appearing in Eq. (2) is $\gamma \approx 0.77$ [6]. Moreover, as one can see in Fig. 3 the basins of attraction of both attractors just before the crisis have fractal boundaries. Our aim is to control the intermittent dynamics, i.e., to force the system to remain on one of the former attractors for an arbitrarily long time using small, occasional perturbation of the accessible parameter.

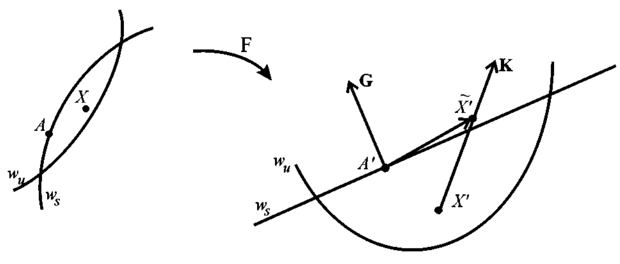

FIG. 4. Escape region and its image after one iteration, illustration of the control procedure (see text).

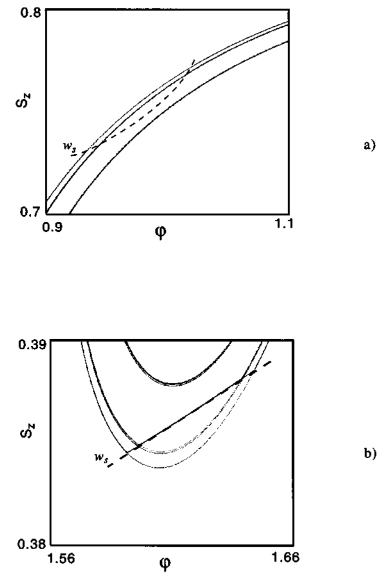

FIG. 5. (a) A fragment of the attractor and the stable manifold of the mediating orbit (dashed line) limiting the escape region R after crisis [$B = 1.005$, other parameter values as in Fig. 2(a)]. (b) The image of the region shown in (a) after the next iteration with the control applied. The points that would otherwise form the subsequent escape region (below w_s) are shifted to the other side of w_s.

III. ALGORITHM OF CONTROL

Let us put the control algorithm more generally, for a system described by a 2D map

$$\mathbf{X}' = \mathbf{F}(\mathbf{X}, p) \tag{9}$$

depending on an accessible parameter p. For simplicity we take into account a homoclinic crisis, but the considerations below can be easily extended to the heteroclinic case. Imagine that we have a chaotic attractor in the closure of the unstable manifold w_u of a saddle periodic orbit. The orbit lies at the border of the basin of attraction of the attractor, and the border, in turn, is in the closure of the stable manifold w_s of the orbit. When the parameter p reaches the crisis value p_c the unstable manifold touches the stable one, i.e., infinitely many homoclinic points appear. As p is further increased, cross-hatched regions are formed, which cover almost the whole area of the former attractor and its basin of attraction. A sequence of these regions, as they are iterated one onto another, determine the way of escape of the system from the attractor.

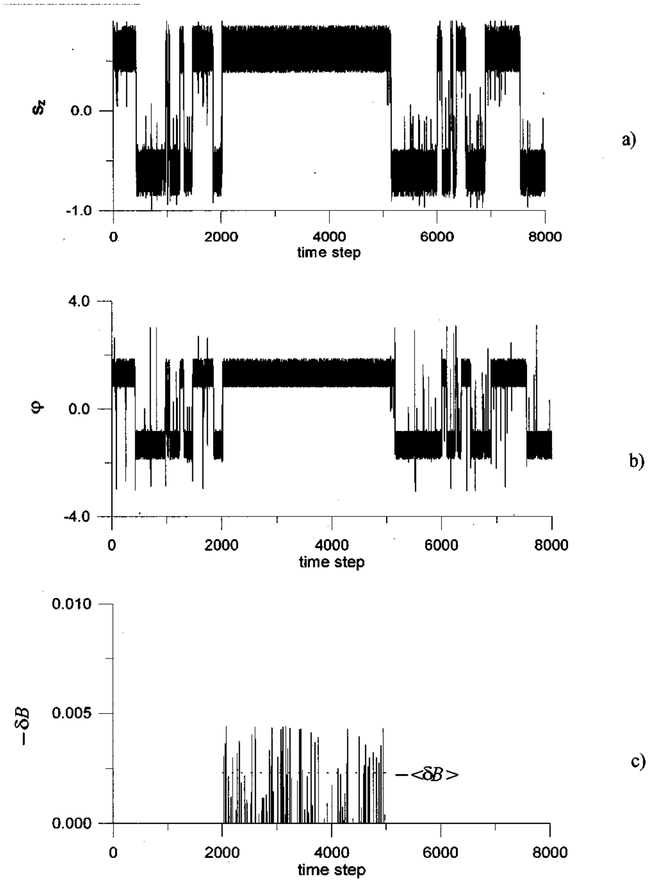

FIG. 6. (a), (b) show the time evolution of map (8), (c) shows the respective changes of the parameter B. Parameter values as in Fig. 2(a). The control was turned on at $t=2000$ and turned off at $t=5000$. The dashed line in (c) denotes the average value of δB calculated over $\delta B \neq 0$.

In order to preserve the motion on the former attractor, destructed or enlarged by the crisis we choose, out of the series mentioned above, a pair of subsequent escape regions, close to the mediating orbit, so that in the second region w_s can be approximated by a straight line and w_u by a parabola (Fig. 4). At every time step we check if the evolving system enters the first of the regions. If so, we apply a small perturbation of a parameter to shift the system out of the escape region in the next iteration. In fact, we only need to check on which side of w_s the phase point is, because the attractor is naturally limited by w_u. We choose a point $\mathbf{A} \in w_s$ and assume that the first escape region is small. Using the linear approximation of the dynamics (9) in this region we have (see Fig. 4)

$$\tilde{\mathbf{X}}' = \mathbf{F}(\mathbf{X}, p+\delta p) \approx \mathbf{F}(\mathbf{X}, p) + \mathbf{K}(\mathbf{A}, p)\,\delta p$$

$$= \mathbf{X}' + \mathbf{K}(\mathbf{A}, p)\,\delta p, \quad \mathbf{K}(\mathbf{X}, p) = \frac{\partial \mathbf{F}(\mathbf{X}, p)}{\partial p}. \qquad (10)$$

Denoting a vector perpendicular to w_s and pointing towards the attractor by $\mathbf{G}$ the condition to shift the phase point to the other side of w_s is

$$(\tilde{\mathbf{X}}' - \mathbf{A}') \cdot \mathbf{G} > 0, \qquad (11)$$

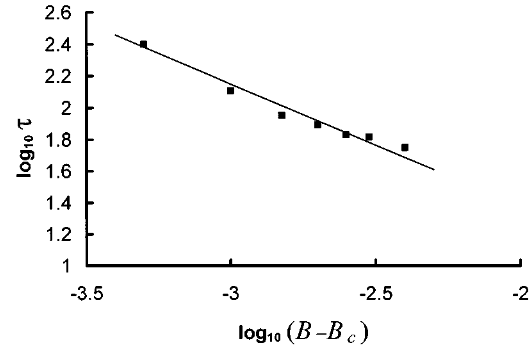

FIG. 7. Mean value of the time between control pulses as the function of the distance $B - B_c$ from the crisis. Solid line shows the dependence obtained from Eq. (2).

where $\mathbf{A}' = \mathbf{F}(\mathbf{A}, p)$. Hence, using Eq. (10) we have

$$\delta p \gtrless -\frac{(\mathbf{X}' - \mathbf{A}') \cdot \mathbf{G}}{\mathbf{K} \cdot \mathbf{G}} \approx -\frac{[\hat{J}_{\mathbf{F}}(\mathbf{A})(\mathbf{X} - \mathbf{A})] \cdot \mathbf{G}}{\mathbf{K} \cdot \mathbf{G}}, \qquad (12)$$

with the sign ‘‘$>$’’ if $\mathbf{K} \cdot \mathbf{G} > 0$ and ‘‘$<$’’ if $\mathbf{K} \cdot \mathbf{G} < 0$. The first case appears if the parameter p decreases reaching the crisis value p_c ($\delta p > 0$) and the other one if the parameter increases ($\delta p < 0$). $\hat{J}_F$ is the Jacobian matrix of the map (9). Obviously, the algorithm fails if $\mathbf{K} \cdot \mathbf{G} = 0$.

The parameter p will be perturbed according to Eq. (12) every time the system tries to escape from our controlled part of the attractor, thus the time between two subsequent $\delta p \neq 0$ has the distribution (1) with τ determined by Eq. (2).

We can also calculate the mean value of the minimal perturbation required to obtain control. To do this we assume a simplified case when $w_u = \{(x,y): y = ax^2\}$, $w_s = \{(x,y): y = Y\}$ and take $\mathbf{A} = (0,Y)$, $\mathbf{K} = [l,k]$, and $\mathbf{G} = [0,1]$ in (12). Then, putting equality in Eq. (12) one gets

$$\delta p = -\frac{(\mathbf{F}(\mathbf{X}) - \mathbf{A}) \cdot \mathbf{G}}{\mathbf{K} \cdot \mathbf{G}} \qquad (13)$$

and the mean value

$$\langle \delta p \rangle = \frac{Y}{k} - \frac{\langle y \rangle}{k}. \qquad (14)$$

The average $\langle y \rangle$ can be calculated using different assumptions concerning the invariant measure of the attractor within the escape region. We consider two cases: (i) when the points of the attractor are uniformly distributed along w_u (on the border of the escape region), or (ii) within the whole escape region. Using the linear approximation $Y \approx m(p - p_c)$ we get, after a proper integration,

$$\langle \delta p \rangle \approx \frac{2}{3}\frac{m}{k}(p - p_c) \quad \text{‘‘border average,’’} \qquad (15)$$

$$\langle \delta p \rangle \approx \frac{2}{5}\frac{m}{k}(p - p_c) \quad \text{‘‘area average.’’} \qquad (16)$$

The result (15) has been obtained using the approximation

$$\int_0^{\sqrt{Y/a}} ax^2 \sqrt{1+4x^2}\,dx \Big/ \int_0^{\sqrt{Y/a}} \sqrt{1+4x^2}\,dx \approx Y/3$$

valid for small Y/a.

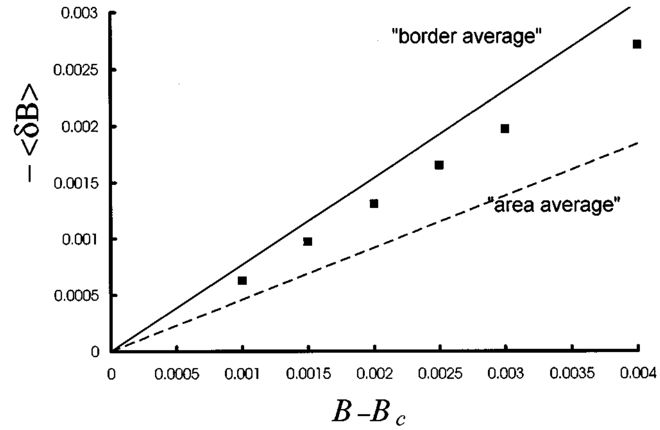

FIG. 8. Mean parameter change $\langle \delta B \rangle$ (calculated over $\delta B=0$) vs the distance from the crisis obtained from Eqs. (15) (solid line) and (16) (dashed line), and computer simulations (points).

IV. NUMERICAL RESULTS

The method described above was applied to control the intermittent behavior emerged in the spin map (8) for $\lambda_c=0.105\,494\,2$, $\tau_c=2\pi$, $A_c=1$, and $B>B_c=1$. As our escape region we have chosen the one arisen from the tangency denoted by R in Fig. 3, its image is denoted by R'. One can see both regions blown up in Fig. 5 after the crisis. Figure 6 shows an example of the controlled dynamics for $B=1.005$. Up to $t=2000$, when the control was turned on, we can see intermittent jumps between both symmetric parts of the attractor. Then, after jumping to the ''upper'' attractor, the system remained there until we switched the control off at $t=5000$ and the intermittent dynamics followed again. The relevant parameter changes, δB, are shown in Fig. 6(c). The control signal $\delta B\neq 0$ was applied, on average, every 39 time steps and its mean value $\langle -\delta B\rangle=0.0023$. If we consider the parameter changes at every time step, including $\delta B=0$ we would obtain another mean value $\langle\!\langle -\delta B\rangle\!\rangle=0.000\,059\ll(B-B_c)$. The control was performed for different values of $B>B_c$ and the mean time τ between subsequent control pulses and their amplitude $\langle \delta B\rangle$ was measured as a function of $B-B_c$. As we expected, τ obeys the scaling low (2) (Fig. 7), but for a given $B-B_c$ it is smaller than the mean time between jumps observed by us without control. This means that some of our control interventions are unnecessary. If we look at Fig. 6 we note that at the uncontrolled dynamics, apart from the jumps between the ''upper'' and ''lower'' phases we have random spikes within the phases that do not lead to the jumps. There are no such spikes in the controlled dynamics, so our control prevents both the interphase jumps and the spikes within the phase. This is the result of a fractal structure of the basin of attraction (see Sec. V). Figure 8 shows the mean amplitude of the control signal $\langle \delta B\rangle$ compared to the calculations based on Eqs. (16) and (15). The actual values lie between both extreme lines because, as one can see in Fig. 3, the attractor is distributed neither over the whole escape region nor along its border, but forms a (fractal) series of loops.

V. SUMMARY AND DISCUSSION

The method presented in this paper enables us to keep a chaotic system on a desired part of a chaotic attractor using occasional control pulses proportional to the distance (in parameter space) from the crisis $p-p_c$; the mean frequency of the pulses is $1/\tau$, where τ scales as $(p-p_c)^{-\gamma}$ for small $p-p_c$. The parameter changes are typically greater than those in the method described in [7,8], but we do not have to know the whole (arbitrary long) chaotic orbit to be stabilized together with the linear properties of the system around each point of the orbit.

The control procedure can be improved to achieve smaller parameter changes needed to perform it. First, one can try to minimize the coefficient m/k in Eqs. (16) and (15). It depends on which pair of escape regions we choose to control. Generically, one should iterate the controlled escape region backwards as far as possible, because then m shrinks exponentially: $m\propto|\lambda_s|^n$, where λ_s is the stable eigenvalue of the mediating orbit. But, on the other hand, the region grows exponentially (as $|\lambda_u|^n$, λ_u is the stable eigenvalue of the mediating orbit) in the other direction, so that it undergoes a series of stretchings and foldings and becomes fractal as $n\rightarrow\infty$, and it is more and more difficult to control such an area. Moreover, our approximation of w_s by a straight line and w_u by a parabola (as in Fig. 4) is valid only for a few preimages of the escape region, close to the mediating orbit. Thus, if one chooses an earlier preimage, the method of control should be modified and Eq. (12) will not be valid.

Another way is to make use of the fractal structure of the basins of attraction (Fig. 3). In fact, not every point of our escape region maps outside the desired part of the attractor in a few time steps. The points being directly attracted to the other part form a fractal set within the region. Thus, if we knew the location of the set, we could just move the system away from it with a much smaller parameter change, instead of a shift out of the entire control region. Moreover, the mean frequency of the control pulses would also decrease, as, because of the structure of our escape region, some of the interventions are unnecessary and can be omitted.

ACKNOWLEDGMENT

This work has been supported by the Polish Committee for Scientific Research (KBN), Grant No. 803/P03/95/09.

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